EQF-Note 2013-02-09

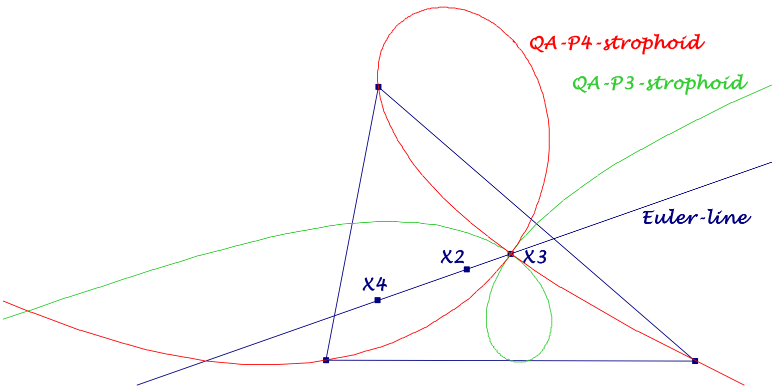
Background for these notes is:
Chris van Tienhoven: Encyclopedia of Quadri-Figures
http://chrisvantienhoven.nl/

Two Strophoids for the Euler Line
(Quadrangle Geometry for Triangles)

Chris van Tienhovens "Encyclopedia of Quadri Figures" EQF contains a lot of points QA-Pn for quadrangles. Taking one of these points, we can consider a mapping

$$P \rightarrow QA-Pn \quad \textit{for} \quad A, B, C, P$$

for points X wrt a reference triangle ABC. Here these QA-points will be QA-P3 (Gergonne-Steiner Point) and QA-P4 (Isogonal Center). The corresponding mappings are applied to the Euler-line. – Barycentric coordinates are used wrt the reference triangle.



Preliminary Remarks

This paper is an example for doing triangle geometry with quadrangle properties, an idea of *Seiichi Kirikami* (personal note). Taking the point to be mapped as fourth point of a quadrangle wrt a reference triangle, we can consider a *QA*-point as image. For the Isogonal Center *QA-P4* here are some examples:

$$X1 \rightarrow X36 \ , \ X2 \rightarrow X187 \ , \ X3 \rightarrow X186,$$
$$X5 \rightarrow X1157 \ , \ X6 \rightarrow X23 \ , \ X7 \rightarrow X1155,$$
$$X8 \rightarrow X1319 \ , \ X9 \rightarrow X2078 \ , \ X10 \rightarrow X1326.$$

In this paper we shall map the Euler-line:

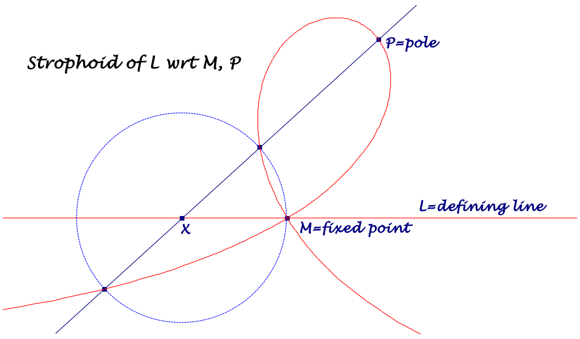
The *QA-P1*-mapping (wrt the *QA*-Centroid) is homothetic with center *X2* and factor *1/4* and maps the Euler-line into itself.

The *QA-P2*-mapping (wrt the Euler-Poncelet Point) maps the Euler-line into the nine-point circle of the reference triangle.

The *QA-P3*- and the *QA-P4*-mappings give strophoids for the Euler-line.

Strophoids

We shall consider a strophoid as defined by *E. H. Lockwood* [1]. Let L be a line (*defining line*), M a point (*fixed point*) on L and P another point (*pole*) not on L, then the strophoid of L wrt M and P is the locus of the intersections of the lines XP – X point on L – with circles round X through M.



The fixed point M on L is a double point with orthogonal tangents. Reflecting the strophoid in a circle round M, there is an orthogonal hyperbola (more properties see my homepage: 06-2, 07-3).

***QA-P3*-Strophoid for the Euler-Line**

The Euler-line has the equation

$$\sum_{cycl}(b^2-c^2)S_A x = 0$$

and *QA-P3* for *A, B, C, P(u:v:w)* has the coordinates (only the first coordinate is specified):

$$(S_A u^2 + S_B v^2 + S_C w^2 - (u+v+w)(-S_A u + S_B v + S_C w))$$
$$(2a^2 uvw - (-a^2 vw + b^2 wu + c^2 uv)(v+w)) \ .$$

Then the image of the Euler-line is a cubic with the equation:

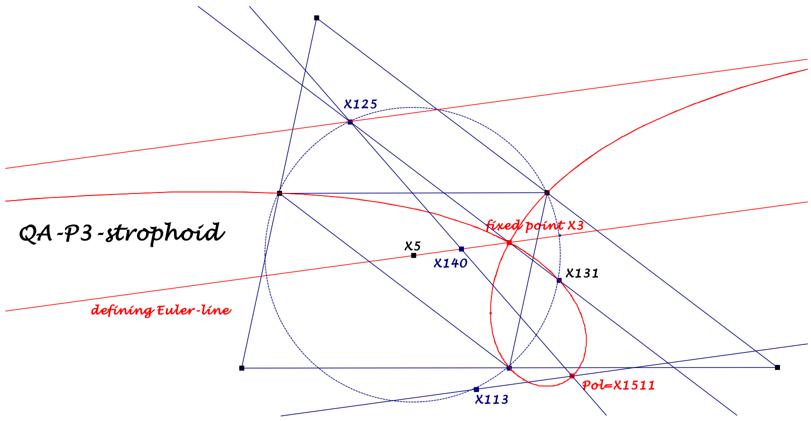
$$4(S_A - S_B)(S_B - S_C)(S_C - S_A)xyz$$
$$-\sum_{cycl}(S_B - S_C)(b^2c^2x^3 - b^2(3S_A - S_B)x^2z - c^2(3S_A - S_C)x^2y) = 0$$

This cubic is a strophoid:

- Defining line is the Euler-line.
- Fixed point is the orthocenter *X3*.

- Pole is the image of the orthocenter *X3* wrt the *QA-P3*-mapping, that is *X1511* in *ETC*:

  $$(a^2(S^2-3S_A^2)(S^2-3S_BS_C):...)$$



Further properties:

- The strophoid is circumscribed the medial triangle.
- The asymptote is a parallel to the Euler-line through *X125*.
- The pole is the reflection of *X125* in *X140*.
- The fourth intersection of the strophoid and the circumcircle of the medial triangle lies in *X131*, which is the image wrt the *QA-P3*-mapping of the reflection of *X3* in *X5*.
- The orthogonal tangents in the fixed point *X3* are parallel to the asymptotes of the Jerabek-hyperbola (which is the isogonal conjugate of the Euler-line).
- If we consider an orthogonal hyperbola, centered in the middle of *X3* and the pole with asymptotes parallel to those of the Jerabek-hyperbola, the reflection in a circle round *X3* through the pole will give the strophoid.

QA-P4-Strophoid for the Euler-Line

QA-P4 for *A, B, C, P(u:v:w)* has the coordinates (only the first coordinate is specified):

$$a^2vw(a^2vw+(a^2-b^2)uv+(a^2-c^2)uw-2S_Au^2 .$$

Then the image of the Euler-line is a cubic with the equation:

$$a^2b^2c^2(a^2-b^2)(b^2-c^2)(c^2-a^2)xyz$$

$$+\sum_{cycl}(b^4c^2S_B(4S_BS_C-a^2b^2)x^2z-b^2c^4S_C(4S_BS_C-a^2c^2)x^2y)=0 .$$

This cubic is a strophoid:

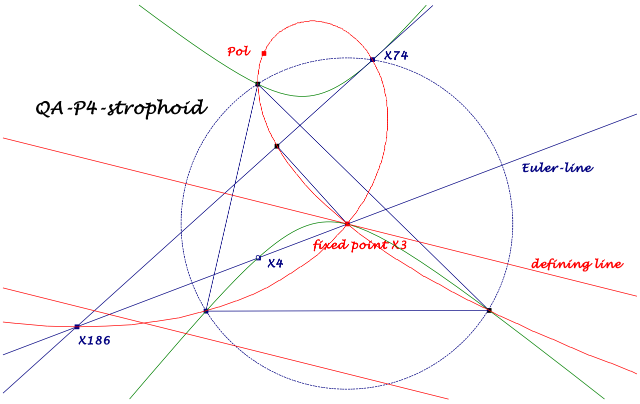
- Defining line is the tangent in the circumcenter *X3* to the Jerabek-hyperbola (see above).

  $$\sum_{cycl} b^2c^2(b^2-c^2)S_BS_C x=0$$

- Fixed point is the circumcenter *X3*.

- Pole is the image of *X186* wrt the *QA-P4*-mapping , not in *ETC*. *X186* is the reflection of the orthocenter *X4* in the circumcircle.

$$pole = \left(\frac{a^2 S_A (S^2 - S_A^2)}{S^2 - 3S_A^2} : \frac{b^2 S_B (S^2 - S_B^2)}{S^2 - 3S_B^2} : \frac{c^2 S_C (S^2 - S_C^2)}{S^2 - 3S_C^2}\right)$$



Further properties:

- The strophoid contains the vertices of the reference triangle.
- The strophoid is the reflection of the Jerabek-hyperbola in the circumcircle.
- The *QA-P4*-mapping of the point at infinity of the Euler-line gives *X74* as a point on the strophoid, the fourth intersection of the circumcircle and the Jerabek-hyperbola.
- *X186* is a point on the strophoid as image of *X3* wrt the *QA-P4*-mapping.
- The asymptote is a parallel to the defining line through the reflection of the pole in the fixed point.
- The pedal point on *X74.X186* wrt the circumcenter *X3* is a point on the strophoid (not in *ETC*).

Final remark to the first figure: The two strophoids have a common fixed point in the circumcenter *X3* and in this point the same orthogonal tangents parallel to the asymptotes of the Jerabek-hyperbola. There exists a further common point *X187*, which is the reflection of the Lemoine-point *X6* in the circumcircle. *X187* is the image of the centroid *X2* by the *QA-P4*-mapping and the *QA-P3*-image of the reflection of *X2* in the circumcircle.

References:

[1] E. H. Lockwood: A Book of Curves. – Cambridge, At the University Press 1961.

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