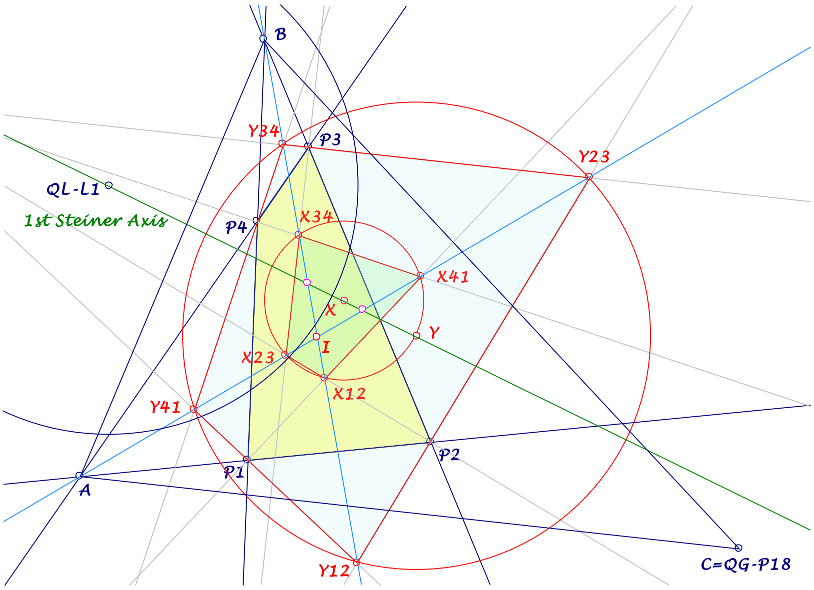
EQF-Note 2014-08-16

Background for these notes is:
Chris van Tienhoven: Encyclopedia of Quadri-Figures
http://www.chrisvantienhoven.nl/

Angle Bisectors of a Quadrigon

It is well-known, that the angle bisectors of a quadrigon give a cyclic quadrigon. Here is described the EQF-geometry of this constellation with calculations in barycentric coordinates. Reference triangle is the Quasi Isogonal Triangle QG-Tr3 (see QFG-messages 343 and 345).



The Reference Triangle

Here an unusual reference triangle is used: *QG-Tr3* with vertices *A* and *B* in the intersections of opposite *QG*-sidelines and the Quasi Isogonal Crosspoint *C=QG-P18* (see *QFG*-message 343).
A quadrigon then is defined by one vertex such as $P_4(u:v:w)$:

P_2 is the isogonal conjugate of P_4 wrt *QG-Tr3*,
P_1 is the intersection of $A.P_4$ and $B.P_2$,
P_3 is the intersection of $A.P_2$ and $B.P_4$.

In this way the quadrigon gets the vertices:

$$P_1 = (a^2w^2 : c^2uv : c^2uw), \quad P_2 = (a^2vw : b^2wu : c^2uv),$$
$$P_3 = (c^2uv : b^2w^2 : c^2vw), \quad P_4 = (u : v : w).$$

Using the abbreviations

$$A = c^2v^2 + S_A vw + b^2w^2, \quad B = c^2u^2 + 2S_B uw + a^2w^2 \ ,$$

the Miquel Point *QL-P1* has the coordinates

$$QL-P1 = (a^2vB(uA+vB) : b^2uA(uA+vB) : -c^2uvAB)$$

and the 1st Steiner Axis (see *QL-Tf1*) gets the equation

$$-bc^2uAx + ac^2vBy - ab(a-b)(uA+vB)z = 0\,.$$

The Schmidt Circle (see *QL-Tf1*) has the equation

$$\begin{aligned}&c^2(buA-avB)(buAx^2-avBy^2)\\&+ab(ab(uA+vB)^2-c^2uvAB)z^2\\&-2c^2uvAB(S_Cxy+ab(xy+yz+zx))\\&+2(uA+vB)(b^2uS_BAx+a^2vS_ABy)z=0\end{aligned}$$

and the fixed points *QL-2P3* of the Clawson-Schmidt Conjugate *QL-Tf1*, which are the intersections of the Schmidt Circle and the 1st Steiner Axis, have the coordinates

$$(-a\pm\frac{a\sqrt{2vB(ab-S_C)}}{(a-b)\sqrt{uA}}:b\pm\frac{b\sqrt{2uA(ab-S_C)}}{(a-b)\sqrt{vB}}:\frac{c^2}{a-b}).$$

Also the Clawson- Schmidt Conjugate (*CSC*) is acceptable:

$$\begin{aligned}&(x:y:z)\quad\to\\&(a^2vB(uA(c^2xy+(a^2-b^2)yz-b^2z^2)-vB(c^2y^2+2S_Ayz+b^2z^2)\\&:(b^2uA(vB(c^2xy-(a^2-b^2)xz-a^2z^2)-uA(c^2x^2+2S_Bxz+a^2z^2)\\&:-c^2uvAB(a^2yz+b^2zx+c^2xy))\ .\end{aligned}$$

The Two Cyclic Quadrigons

The inner / outer angle bisectors of the quadrigon $P_1P_2P_3P_4$ give two cyclic quadrigons with vertices in the incenters and excenters of the *QL*-triangle components (excenters wrt *A*, *B*):

$$\begin{aligned}&X_{12}=In(P_1,P_2,B),\quad Y_{12}=Ex(P_1,P_2,B),\\&X_{23}=In(P_2,P_3,A),\quad Y_{23}=Ex(P_2,P_3,A),\\&X_{34}=Ex(P_3,P_4,B),\quad Y_{34}=In(P_3,P_4,B),\\&X_{41}=Ex(P_4,P_1,A),\quad Y_{41}=In(P_4,P_1,A),\end{aligned}$$

with coordinates:

$$\begin{aligned}&X_{12},Y_{12}=(a(aw\sqrt{A}\mp(cv-bw)\sqrt{B}):bcu\sqrt{A}:c^2u\sqrt{A}),\\&X_{23},Y_{23}=(acv\sqrt{B}:b(bw\sqrt{B}\mp(cu-aw)\sqrt{A}):c^2v\sqrt{B}),\\&X_{34},Y_{34}=(cu\sqrt{A}\pm(cv-bw)\sqrt{B}:bw\sqrt{A}:cw\sqrt{A}),\\&X_{41},Y_{41}=(aw\sqrt{B}:cv\sqrt{B}\pm(cu-aw)\sqrt{A}:cw\sqrt{B})\ .\end{aligned}$$

(1) The two cyclic quadrigons have common diagonals.

These common diagonals are angle bisectors of opposite sidelines of the reference quadrigon, intersecting in the incenter $I(a:b:c)$ of the reference triangle *QG-Tr3*.

(2) The two cyclic quadrigons have centers on the 1st Steiner Axis.

These centers are

$$\begin{aligned}&X,Y=(\mp acv(-a+b+c)(cu-aw)\sqrt{A}B\\&-a(a-b+c)(cv-bw)\sqrt{B}((a-b)(c^2uv(u+v+w)+w(a^2vw+b^2wu+c^2uv))-cvB)\\&:bcu(a-b+c)(cv-bw)A\sqrt{B}\end{aligned}$$

$$\pm b(-a+b+c)(cu-aw)\sqrt{A}((b-a)(c^2uv(u+v+w)+w(a^2vw+b^2wu+c^2uv))-cuA)$$
$$:\mp c^2v(-a+b+c)(cu-aw)\sqrt{A}B+c^2u(a-b+c)(cv-bw)A\sqrt{B})$$

(3) The two cyclic quadrigons have circumcircles perpendicular to the Schmidt Circle. They are *CSC*-invariant. The radical axis is the 2nd Steiner Axis.

(4) The homothetic centers of the two circumcircles are the intersections of the common diagonals and the 1st Steiner Axis.

These two centers are *CSC*-partners with the coordinates:

$$(-a((a-b)uA-(-a+b+c)vB : bcuA : c^2uA),$$
$$(acvB : b((a-b)vB+(a-b+c)uA : c^2vB).$$

They lie harmonic with the midpoints of the circumcircles of the cyclic quadrigons.

(5) The 3rd diagonals of the cyclic quadrigons intersect in the excenter I_c of the reference triangle.

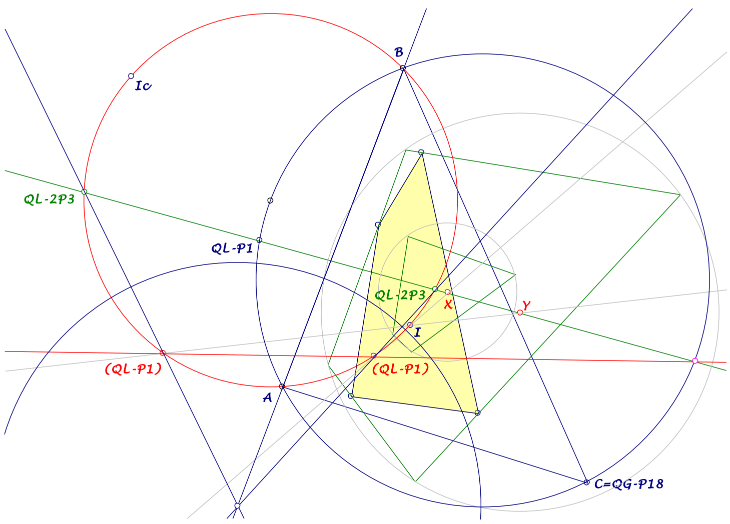
(6) The Morley Points (*QL-P2)* of the cyclic quadrigons lie on the Newton Line *QL-L1* of the reference quadrigon.

(7) The Miquel Points (*QL-P1*) of the cyclic quadrigons lie concyclic …
… with the Schmidt Pair of Points *QL-2P3,*
… with the vertices *QG-2P2* of *QG-Tr3,*
… with the incenter and the excenter *Ic* of *QG-Tr3* as diametral points.

This interesting *QG*-circle is *CSC*-invariant with the equation:

$$c^2xy+(a-b)(ay-bx)z-abz^2=0.$$

The circle is also perpendicular to the two circumcircles of the cyclic quadrigons.



(8) The Miquel Points of the cyclic quadrigons lie collinear with the second intersection of the 1st Steiner Axis and the circumcircle of *QG-Tr3* (midpoint of the excenters I_a and I_b).

The Miquel Points are the second intersections of *XI* and *YI* with this circle under (7).
The coordinates of the Miquel Points are:

$$(a\sqrt{B}(cv-bw)(-(a-b)\sqrt{A}(cu-aw)\mp(a-b+c)\sqrt{B}(cv-bw))$$
$$:b\sqrt{A}(cu-aw)((a-b)\sqrt{B}(cv-bw)\mp(-a+b+c)\sqrt{A}(cu-aw))$$
$$:\sqrt{A}\sqrt{B}c^2(cu-aw)(cv-bw)).$$

There is an inversion, changing the Schmidt Pair of Points *QL-2P3* and these Miquel Points. The corresponding circle is centered on the 3rd diagonal of the reference quadrigon in

$$(a\sqrt{a-b+c})B\sqrt{v}(cv-bw):b\sqrt{-a+b+c})A\sqrt{u}(cu-aw):0).$$

(9) The centers *X, Y* of the cyclic quadrigons are the Double Points of a *QA*-Line Involution *QA-Tf1* on the 1st Steiner Axis.

1st pair of points:

intersections *S* and *S´* of the 1st Steiner Axis and the angle bisectors of *QG-Tr3* at *A* and *B* (see (1), (4)).

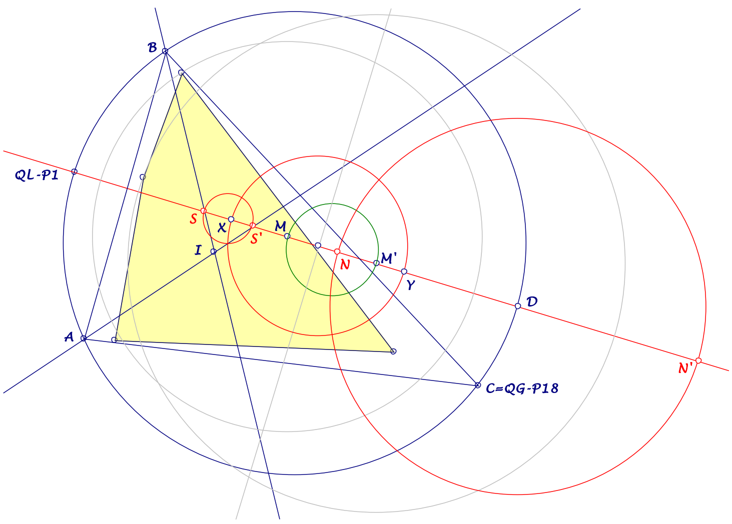
2nd pair of points:

… let *M* and *M´* be the midpoints of circles centered on the 1st Steiner Axis and containing opposite vertices of the quadrigon,

… let *D* be the second intersection of the 1st Steiner Axis and the circumcircle of *QG-Tr3* (see (8)),

… let *Ci* be a circle round *D* orthogonal to the Thales circle of *MM´*,

… the intersections *N* and *N´* of *Ci* and the 1st Steiner Axis are the 2nd pair of points.



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