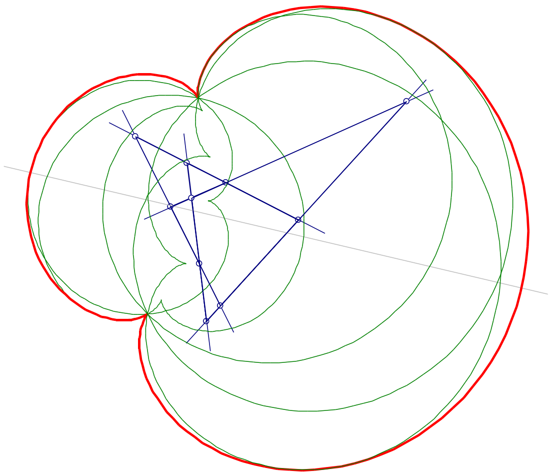
EQF-Note 2014-11-23

Background for these notes is:
Chris van Tienhoven: Encyclopedia of Quadri-Figures
http://www.chrisvantienhoven.nl/

Morley's Tetracardioid of a 5-Line
(Construction and Equation)

Morley describes for a 5-line a tetracardioid C4, contacting the 5 cardioids QL-Qu1 of the 4-line-components [1]. Here is given a construction of this curve and its equation.



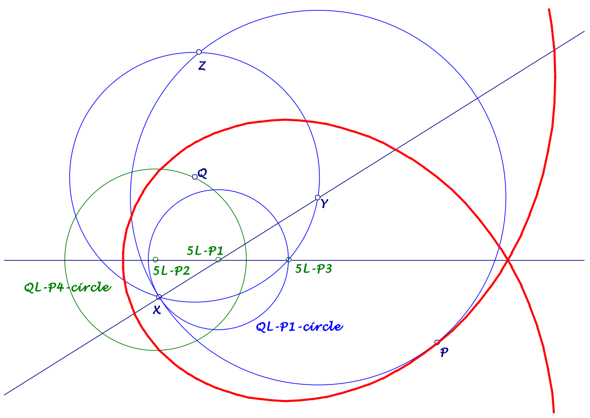
Construction

The tetracardioid is determined by a point and a circle: The point is *5L-P1*, the center of the 5 concentric Miquel points *QL-P1* of the 4-L. The circle is the circumcircle of the 5 concentric Miquel circumcenters *QL-P4* of the 4-L with center *5L-P2*. The inverse of *5L-P1* wrt the *QL-P4*-circle is the point *5L-P3*. The circle round *5L-P1* through *5L-P3* is the *QL-P1*-circle (see *QFG* #710).

- **The tetracardioid of a 5-Line is determined by the point *5L-P1* and the *QL-P4*-circle.**

Construction (see *QFG*-message 815):

... Let *Q* be a variable point on the *QL-P4*-circle,
... let *Ci1* be a circle round *Q* through *5L-P3*,
... let *X* be the second intersection of *Ci1* and the *QL-P1*-circle,
... let *Y* be the second intersection of *X.5L-P1* and *Ci1*,
... let *Ci2* be a circle round *Y* through *X*,
... let *Z* be the second intersection of *Ci1* and *Ci2*,
... then *Z* reflected in *Y* is a point *P* of Morley's tetracardioid.



- **The tetracardioid can be a limaçon-like curve or a curve with two real cusps.**

Equation

We use Cartesian coordinates:

5L-P2(0,0), 5L-P1(0,p), r radius of the QL-P4-circle.

Following the construction we get the equation:

$$4 p r^4 (16 p^3 r^2 - 9 p r^4 - 54 p^2 r^2 y + 27 r^4 y + 27 p^3 y^2) - 3 r^4 (16 p^4 - 60 p^2 r^2 + 27 r^4 + 36 p^3 y) (x^2 + y^2) + 6 p^2 r^2 (2 p^2 + 3 r^2) (x^2 + y^2)^2 + p^4 (x^2 + y^2)^3 = 0$$

Discussion:

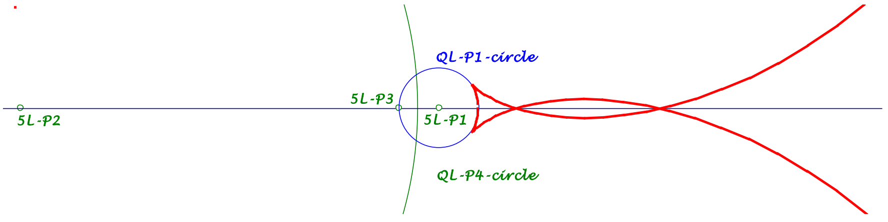
- **The tetracardioid is limaçon-like, if *5L-P1* lies inside the *QL-P4*-circle ($p \le r$).**

Node $N(0, \dfrac{3r^2 + r\sqrt{9r^2 - 8p^2}}{2p})$.

Vertices $V_1(0, \dfrac{-4pr - 3r^2}{p})$ and $V_2(0, \dfrac{4pr - 3r^2}{p})$.

The contact points of the double-tangent $T_{1,2}(\pm 2r, \dfrac{3r^2}{p})$.

- **The tetracardioid has two cusps, if *5L-P1* lies outside the *QL-P4*-circle ($p > r$).**
- **The cusps lie on the *QL-P1*-circle.**



There can be two nodes, not always real:

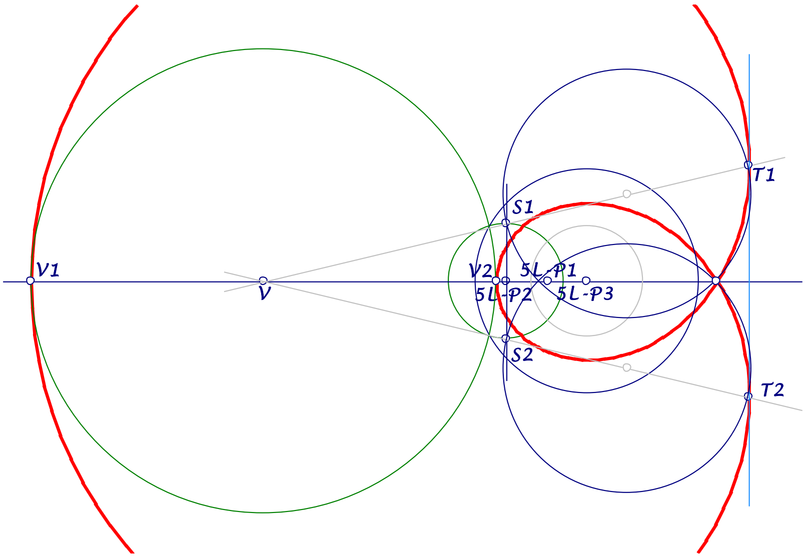
$$N_{1,2}(0,\frac{3r^2 \pm r\sqrt{9r^2-8p^2}}{2p})$$

Vertices (see above) $V_1(0,\frac{-4pr-3r^2}{p})$ and $V_2(0,\frac{4pr-3r^2}{p})$.

Cusps $C_{1,2}(\pm\frac{2r}{p^3}\sqrt{(p^2-r^2)^3},\frac{r^2(3p^2-2r^2)}{p^3})$.

Construction of special points

… Take a circle round *5L-P3*, so that *5L-P1* and *5L-P2* lie inverse and double the radius for a further concentric circle,
… wrt this circle draw the inverse of the *QL-P4*-circle,
… the last circle cuts the line *5L-P1.5L-P2* in the vertices V_1, V_2 of the tetracardioid.
… Consider the intersections S_1, S_2 of the *QL-P4*-circle and a perpendicular line through *5L-P2* wrt *5L-P1.5L-P2*,
… reflect the midpoint *V* of the vertices in S_1 and S_2,
… these are the contact points T_1, T_2 of the double-tangent.
… The Thales circles about S_iT_i contain the nodes.



[1] F. Morley: On Reflexive Geometry, Transactions of the American Mathematical Society, Volume 8, 1907.

Eckart Schmidt
http://eckartschmidt.de
eckart_schmidt@t-online.de