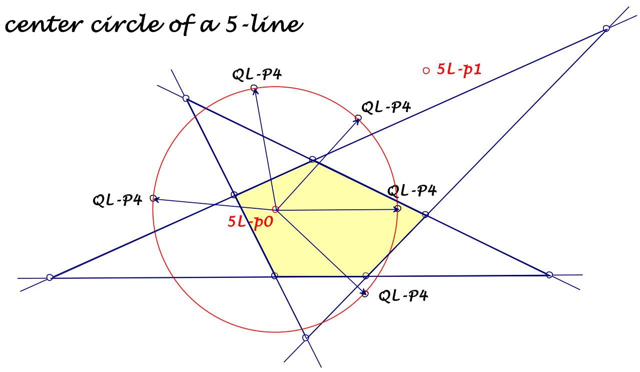
EQF-Note 2014-12-10

Background for these notes is:
Chris van Tienhoven: Encyclopedia of Quadri-Figures
http://www.chrisvantienhoven.nl/

Morley Points for n-Lines

Morley tried, to generalise the triangle orthocenter for n-lines. In his paper "Orthocentric Properties of the Plane n-Line"(1902) we find ambitious analytic calculations for special points and circles, but no explicit constructions. Here is a summary of results out of a discussion in QFG with Bernard Keizer and Chris van Tienhoven.



Center Circle and Morley Point p_0

Beginning with a 3-line we have the circumcircle and its center as Morley point $3L\text{-}p_0$. For a 4-line we get − omitting one line − four concyclic $3L\text{-}p_0$. The corresponding circle will be the center circle of the 4-line – that is the Miquel Circle *QL-Ci3* – and its midpoint $4L\text{-}p_0$ = *QL-P4*.
In this way we get a recursive definition for the Morley point $nL\text{-}p_0$ as center of the *n* concyclic $(n\text{-}1)L\text{-}p_0$ of the n-line, omitting one line.

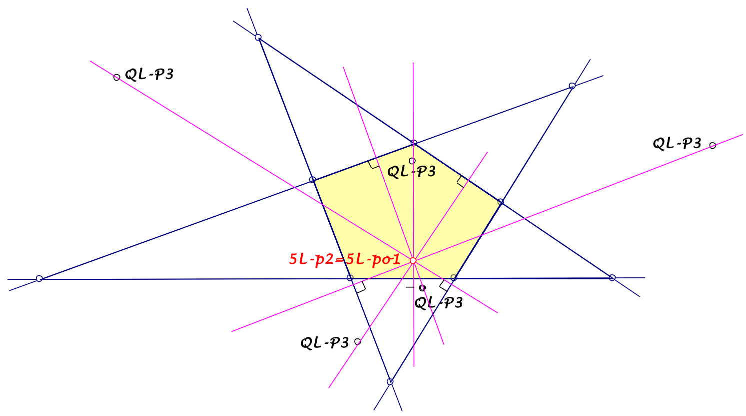
Morley point p_1

Adding the vectors with origin $nL\text{-}p_0$ and extremities $(n\text{-}1)L\text{-}p_0$ we get the Morley point $nL\text{-}p_1$. For a 3-line $3L\text{-}p_1$ is the orthocenter and for a 4-line we find $4L\text{-}p_1$ = *QL-P3*.

Morley point p_2

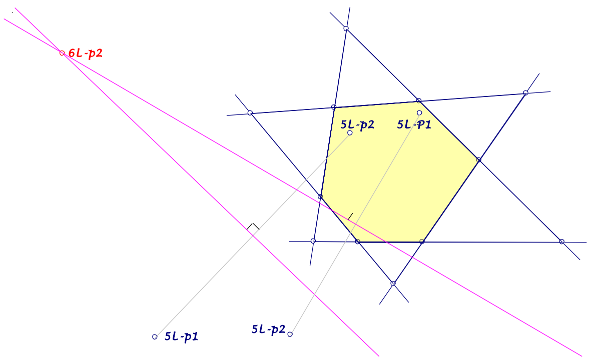
We give a recursive definition: Let *nL-px* be the centroid of the *n* points $(n\text{-}1)L\text{-}p_1$ and divide $nL\text{-}p_1.nL\text{-}px$ in the ratio $-n/(n\text{-}2)$, then you get $nL\text{-}p_2$. For a 3-line let $3L\text{-}p_2$ be the orthocenter. For a 4-line we get $4L\text{-}p_2$ as reflection of *QL-P3* in *QL-P2*. For a 5-

line $5L\text{-}p_2$ is the common point of the perpendiculars through $4L\text{-}p_1 = QL\text{-}P3$ wrt the omitting line. For a 6-line $6L\text{-}p_2$ is the common point of the perpendicular bisectors of $5L\text{-}p_1.5L\text{-}p_2$.



Morley points p_i

It seems, that the following recursive definition holds for $nL\text{-}p_{i+1}$: Let $nL\text{-}px$ be the centroid of the n $(n\text{-}1)L\text{-}p_i$ and divide $nL\text{-}p_i.nL\text{-}px$ in the ratio $-n/(n\text{-}i\text{-}1)$.
This can be controlled with Morley's theorem 5, which says, that $nL\text{-}p_i$ has distances with a fixed ratio to $(n\text{-}1)L\text{-}p_i$ and $(n\text{-}1)L\text{-}p_{i-1}$ of the included (n-1)-lines.
As further control Morley mentioned in a remark to theorem 5 for an even number of lines: $2nL\text{-}p_{n-1}$ is equidistant of $(2n\text{-}1)L\text{-}p_{n-1}$ and $(2n\text{-}1)L\text{-}p_{n-2}$.

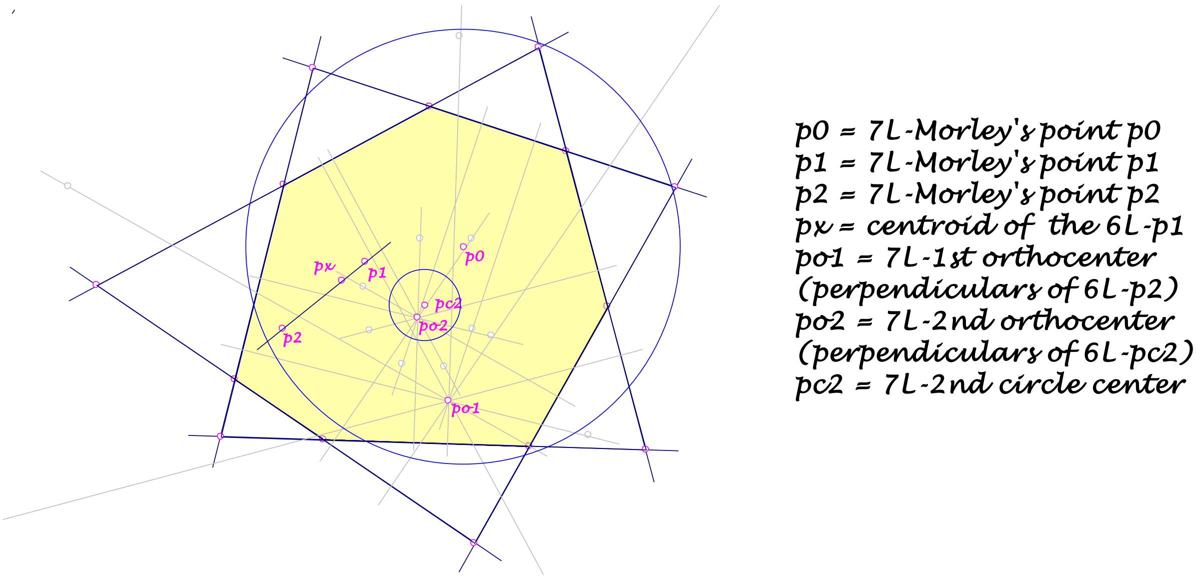


Morley's 1st Orthocenter p_{o1}

Morley's 1st orthocenter is only defined for odd n, beginning with $n=5$. Morley describes this point $nL\text{-}p_{o1}$ as common point of the perpendiculars of $(n\text{-}1)L\text{-}p_2$ wrt the omitting line.
For $n=5$ holds $5L\text{-}p_{o1} = 5L\text{-}p2$, but for $n=7$ these are two different points!
For even n the first orthocenters $(n\text{-}1)L\text{-}p_{o1}$ are collinear on the so called "directrix".

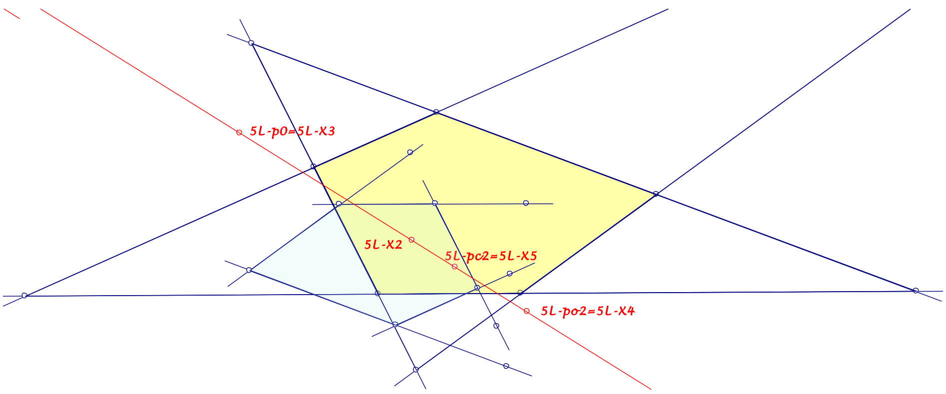
Morley's 2^{nd} Circle Center p_{c2} and 2^{nd} Orthocenter p_{o2}

The 1^{st} circle will be the center circle. For a 3-line Morley's 2^{nd} circle is the nine-point circle with center $X5 = 3L\text{-}p_{c2}$. For a n-line the perpendicular lines through $(n\text{-}1)L\text{-}p_{c2}$ wrt the omitted line will coincide in Morley's 2^{nd} orthocenter $nL\text{-}p_{o2}$. This point – in Morley's paper h – is the external center of similitude for the 1^{st} and 2^{nd} circle (theorem 10). Then the center of the 2^{nd} circle divides $nL\text{-}p_0.nL\text{-}p_{o2}$ in the ratio $(n\text{-}2):1$. The radius of the 2^{nd} circle is $1/(n\text{-}1)$ of the radius of the 1^{st} circle (theorem 9).
For a 4-line holds $4L\text{-}p_{o2} = QL\text{-}P2$ and $4L\text{-}p_{c2}$ divides $QL\text{-}P2.QL\text{-}P4$ with ratio 1:2.



nL-Quasi Euler Line

Morley gave a generalisation of the circumcenter and the orthocenter of a triangle, here is proposed a generalisation of the Euler line:



Let be:

$nL\text{-}X3 = nL\text{-}p_0$ the midpoint of the center circle,

$nL\text{-}X4 = nL\text{-}p_{o2}$ the 2^{nd} orthocenter,

$nL\text{-}X5 = nL\text{-}p_{c2}$ the 2^{nd} circle center.

For $n=3$ holds *3L-Xi* = *Xi* (see *ETC*), for $n=4$ holds: *4L-X3* = *QL-P4*, *4L-X4* = *QL-P2*, *4L-X5* divides *QL-P4.QL-P2* in the ratio *2:1*.

Generalisation: $$\frac{nL-X3.nL-X5}{nL-X5.nL-X4}=\frac{n-2}{1}$$

Now let *nL-X2* be the "nL-quasi-centroid", which is the homothetic center of the reference n-line and the n-line of the parallels to L_i through *(n-1)L-X5*.
For $n=3$ holds *3L-X2* = *X2*, for $n=4$ holds *4L-X2* = *QL-P22*.

Generalisation: $$\frac{nL-X3.nL-X2}{nL-X2.nL-X4}=\frac{n-2}{2}$$

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