# Miquel Points
# and Inscribed Triangles

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*Triangles with vertices on the sidelines of a reference triangle are called inscribed triangles. Relations between inscribed triangles and their Miquel points are subject of this work. ABC-similar triangles, equilateral triangles, isosceles right triangles, but also cevian triangles are considered as inscribed triangles. We discuss up to twelve Miquel points for similar inscribed triangles. – Analytical calculations are made in barycentric coordinates.*

## 1. Miquel points

We begin with some well known geometrical results for inscribed triangles and Miquel points: For an inscribed triangle with vertices $A', B', C'$ on the sidelines $BC, CA, AB$ of a given reference triangle $ABC$ the circumcircles of the residual triangles $AB'C'$, $BC'A'$, $CA'B'$ are concurrent (Fig. 1). The common point is the Miquel point of the inscribed triangle ([2], 44; [4], 79). If $A', B', C'$ are collinear we get the Miquel point of the corresponding complete quadrilateral. The lines from the Miquel point to the vertices of an inscribed triangle are equally inclined to the sides of the given triangle ([2], 45; [4], 83). If $M$ is the Miquel point, we shall denote the angle

$$\varphi = \angle MA'B = \angle MB'C = \angle MC'A$$

the Miquel angle of the inscribed triangle. More important is the fact, that all directly similar inscribed triangles have the same Miquel point ([6], 211). If the inscribed triangle is a pedal triangle of a point $P$, this point must be the Miquel point of course. So we can take the pedal triangle among the considered similar inscribed triangles to get the Miquel point.

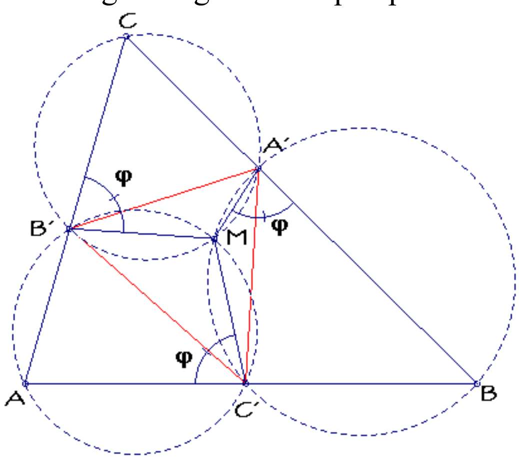


Fig. 1. The Miquel point

## 2. The two Miquel points of similar inscribed triangles

Similarity can be proved by properties of the angles or the ratios of sidelengths. For further calculations in barycentric coordinates we shall prefer the last possibility. Here we give three synthetic remarks for pedal triangles, which are useful for further calculations.

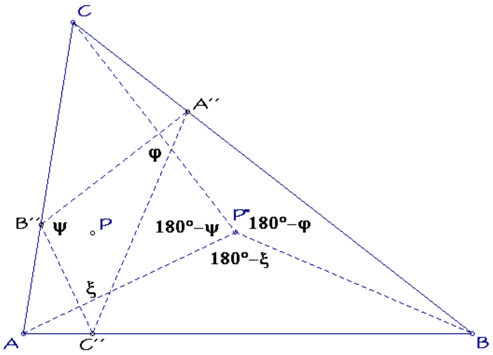


Fig. 2. Orthology of the pedal triangle and the reference triangle

2.1 The pedal triangle $A''B''C''$ of a point $P$ is orthologic [8] to the reference triangle with centers $P$ and its isogonal conjugate $P*$ (see [1], 1210). The angles

$$\angle C''B''A'', \quad \angle A''C''B'', \quad \angle B''A''C''$$

are supplementary or negative or equal to

$$\angle CP*A, \quad \angle AP*B, \quad \angle BP*C$$

if $P$ is a point inside the reference triangle or between the reference triangle and the circumcircle or outside the circumcircle.

An immediate consequence: The Miquel point of equilateral inscribed triangles must be an isodynamic point.

2.2 For the pedal triangle $A''B''C''$ of a point $P$ the quadrilateral $AB''PC''$ has a circumcircle with diameter $AP$. For the second diagonal the law of sines gives

$$B''C'' = a'' = 2r_{AB''C''}\sin A = AP \cdot \sin A = \frac{AP \cdot a}{2r_{ABC}}.$$

If we denote the area of the triangle $ABC$ by $\Delta = \frac{S}{2}$ we have

$$r_{ABC} = \frac{abc}{4\Delta} = \frac{abc}{2S}$$

and $$\frac{a''}{b''} = \frac{a}{b}\frac{AP}{BP}, \quad \frac{b''}{c''} = \frac{b}{c}\frac{BP}{CP}, \quad \frac{c''}{a''} = \frac{c}{a}\frac{CP}{AP}.$$

An immediate consequence: The common points of the Apollonius circles of the reference triangle have isosceles pedal triangles.

2.3 We consider similar inscribed triangles $A'B'C'$ with $A' \in BC$, $B' \in CA$, $C' \in AB$.

**Theorem 1.** **The Miquel points of a pair of inversely similar inscribed triangles are inverses with respect to the circumcircle.**

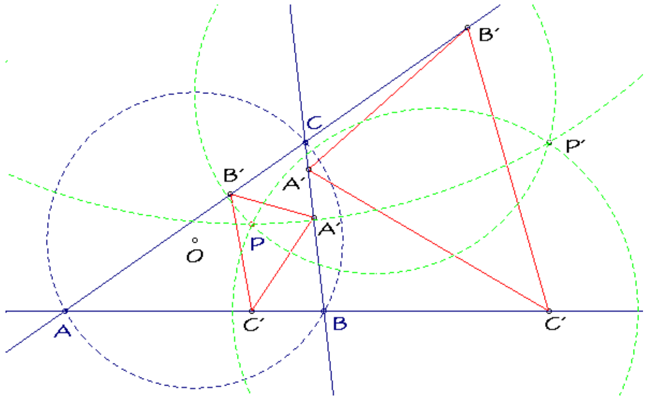


Fig. 3. The two Miquel points of similar inscribed triangles

Proof: For the distances of the Miquel point $P$ to the vertices $A, B, C$ we have for example

$$\frac{AP}{BP}=\frac{a''}{b''}\frac{b}{a}.$$

Therefore the Miquel point $P$ is on the Apollonius circle over $AB$ for this ratio. All circles through $A$ and $B$ are orthogonal to this Apollonius circle, especially the circumcircle of $ABC$. Hence the Apollonius circle is invariant under the inverse in the circumcircle and the inverse $P'$ of the Miquel point $P$ has the same ratios for the distances to the vertices and further on for the sidelengths of its pedal triangle. The pedal triangle of $P'$ is similar to the pedal triangle of $P$ but necessarily with other orientation ([1], S.1214). This proves, that directly and inversely similar inscribed triangles have inverse Miquel points with respect to the circumcircle. □

## 3. Barycentric coordinates

For a point $P$ we consider the pedal triangle $A''B''C''$ and all directly similar inscribed triangles $A'B'C'$ with the Miquel point $P$.

We use barycentric coordinates

$$A(1:0:0),\quad B(0:1:0),\quad C(0:0:1),\quad P(u:v:w)$$

and the Conway notations $S_A, S_B, S_C, S_\omega, S$ with

$$2S_A=-a^2+b^2+c^2,\ \ldots \text{ and } S_\omega=S_A+S_B+S_C$$

$$\text{and } S=\sqrt{S_AS_B+S_BS_C+S_CS_A}=2\Delta.$$

The pedal point

$$C''(c^2u+S_Bw:c^2v+S_Aw:0)$$

divides $AB$ in the ratio

$$\frac{c^2v+S_Aw}{c^2u+S_Bw}$$

and for the sidelengths $a''$ and $b''$ we have

$$\frac{a''}{b''}=\frac{a}{b}\frac{AP}{BP}=\frac{a}{b}\sqrt{\frac{c^2v^2+2S_Avw+b^2w^2}{c^2u^2+2S_Buw+a^2w^2}}.$$

If there are special conditions for similar inscribed triangles, the last equation and its cyclic permutations permit a calculation for the two Miquel points.

Using normalized coordinates, that means dividing the coordinates by the sum of the coordinates, the determinant of the coordinates of $A'', B'', C''$

$$\frac{S^2}{a^2b^2c^2}(a^2vw+b^2wu+c^2uv)$$

gives the ratio of the area of the orientated pedal triangle to the area of the triangle *ABC* [9]. The last factor is the negative power of *P* with respect to the circumcircle, which has the equation

$$a^2yz+b^2zx+c^2xy=0.$$

If *P* lies inside (outside) of the circumcircle, the pedal triangle has the same (contrary) orientation as the reference triangle. If *P* is a point on the circumcircle, the pedal triangle degenerates into the Simson line.

For inscribed triangles $A'B'C'$ directly similar to $A''B''C''$ we take the cotangent of the Miquel angle $\varphi$ as real parameter:

$$\kappa=\cot\varphi \quad \text{with} \quad \varphi=\angle PA'B=\angle PB'C=\angle PC'A.$$

$\kappa=0$ gives the pedal triangle of $P$.
Then the point $C'$ divides *AB* in the ratio

$$\gamma=\frac{c^2v+(S_A+\kappa S)w}{c^2u+(S_B-\kappa S)w}.$$

In the same way we get the ratios $\alpha$ and $\beta$ for *BC* and *CA* by cyclic permutation.
For any real parameter κ we have an inscribed triangle $A'B'C'$ similar to the pedal triangle $A''B''C''$:

$$A'(0:1:\alpha),\ B'(\beta:0:1),\ C'(1:\gamma:0).$$

Conversely: For an inscribed triangle, whose vertices divide the sides in the ratios $\alpha, \beta, \gamma,$ the Miquel point is

$$P(\frac{a^2}{1+\alpha}(\frac{\alpha a^2}{1+\alpha}-\frac{\alpha\beta b^2}{1+\beta}-\frac{c^2}{1+\gamma})$$
$$:\frac{b^2}{1+\beta}(-\frac{a^2}{1+\alpha}+\frac{\beta b^2}{1+\beta}-\frac{\beta\gamma c^2}{1+\gamma})$$
$$:\frac{c^2}{1+\gamma}(-\frac{\alpha\gamma a^2}{1+\alpha}-\frac{b^2}{1+\beta}+\frac{\gamma c^2}{1+\gamma})).$$

These formulas are the background of further calculations.

## 4. *ABC*-similar inscribed triangles

*ABC*-similar means similar to the reference triangle. The medial triangle is directly similar to the reference triangle and is the pedal triangle of the circumcenter. Since the circumcenter has no inverse with respect to the circumcircle there are in view of Theorem 1 no inversely *ABC*-similar triangles.

**Theorem 2.** **The Miquel point of *ABC*-similar inscribed triangles is the common orthocenter which is the circumcenter of the reference triangle.**
**The sidelines envelope parabolas *). These parabolas touch the residuals of the medial triangle. The common focus is the Miquel point and the tangent of the vertex is a sideline of the medial triangle.**

*) For right angled reference triangles this holds only for the hypotenuse.

Proof: Among the *ABC*-similar inscribed triangles the medial triangle is the pedal triangle $A''B''C''$ of the circumcenter

$$O(a^2S_A : b^2S_B : c^2S_C).$$

Then for any real parameter $\kappa$ we obtain an *ABC*-similar inscribed triangle $A'B'C'$:

$$A'(0 : S - S_A\kappa : S + S_A\kappa), \quad B'(S + S_B\kappa : 0 : S - S_B\kappa),$$
$$C'(S - S_C\kappa : S + S_C\kappa : 0)$$

$$\text{with } \frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} = \frac{1}{2}\sqrt{1+\kappa^2},$$

and the common orthocenter is the circumcenter of the reference triangle.
The sidelines for example

$$B'C': \quad -(S - \kappa S_B)(S + \kappa S_C)x + (S - \kappa S_B)(S - \kappa S_C)y + (S + \kappa S_B)(S + \kappa S_C)z = 0$$

envelope a conic with the equation

$$(a^2x - (S_B - S_C)(y - z))^2 - 16S_BS_Cyz = 0$$

touching the residual triangle $AC''B''$ (for an inscribed triangle *XYZ* we call the triangles *AYZ, BZX, CXY* its residuals).

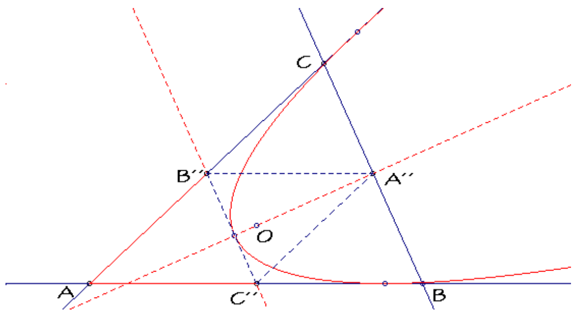

Fig. 5

The only infinite point $F_a(-a^2 : S_C : S_B)$ shows, that the envelope is a parabola with focus on the circumcircle of the residual triangle $AB''C''$. Since the lines $OF_a$ and $B''C''$ are orthogonal, $B''C''$ is the tangent of the vertex, $O$ the focus and $OF_a$ the axis (Fig. 5). □

Remarks: These results can be generalized: For the pedal triangle of a point $P$ the sidelines of similar inscribed triangles envelope parabolas with focus $P$. The tangents of the vertices are the sidelines of the pedal triangle. If $P$ is a point of the circumcircle, the vertices of the pedal triangle are collinear on the Simson line. "Similar" inscribed triangles give lines enveloping the only inscribed parabola of the complete quadrilateral formed by the sidelines of the reference triangle and the Simson line of $P$.

The Euler lines of $ABC$-similar inscribed triangles all pass through the circumcenter of the given triangle which is the common orthocenter of all $ABC$-similar inscribed triangles. If the reference triangle isn´t equilateral, the circumcenters of $ABC$-similar inscribed triangles lie on a perpendicular of the Euler line through the center of the nine-point circle, even as the centroids on a perpendicular of the Euler line through the centroid (see also §6).

## 5. Further *ABC*-similar inscribed triangles

Up to now we considered similarities $A'B'C' \sim ABC$ with $A'$, $B'$, $C'$ on the sides $a$, $b$, $c$. Now we look for the other possibilities of similarity. If for example $A'$, $B'$, $C'$ lie on the sidelines $b$, $a$, $c$, the corresponding Miquel points shall be denoted $M^{\pm}_{bac}$ with respect to the orientation: $M^{+}_{abc}$ is the circumcenter and $M^{-}_{abc}$ doesn't exist.

For the Miquel points $M^{\pm}_{bca}$ and $M^{\pm}_{cab}$ the modified synthetic remark 2.2 gives :

$$M^{\pm}_{bca}: \quad \frac{AP}{BP}=\frac{bc}{a^2}, \quad \frac{BP}{CP}=\frac{ca}{b^2}, \quad \frac{CP}{AP}=\frac{ab}{c^2}$$

$$M^{\pm}_{cab}: \quad \frac{AP}{BP}=\frac{b^2}{ca}, \quad \frac{BP}{CP}=\frac{c^2}{ab}, \quad \frac{CP}{AP}=\frac{a^2}{bc}.$$

These proportions are valid for the Brocard points; their pedal triangles are $ABC$-similar ([2], 62).

**Theorem 3. For *ABC*-similar inscribed triangles the Miquel point $M^{+}_{abc}$ is the circumcenter, $M^{-}_{abc}$ doesn´t exist. The points $M^{+}_{bca}$, $M^{+}_{cab}$ are the Brocard points and $M^{-}_{acb}$, $M^{-}_{cba}$, $M^{-}_{bac}$ the vertices of the second Brocard triangle, all on the Brocard circle. The points $M^{-}_{bca}$, $M^{-}_{cab}$, $M^{+}_{acb}$, $M^{+}_{cba}$, $M^{+}_{bac}$ are the corresponding inverses in the circumcircle, all on the Lemoine line.**

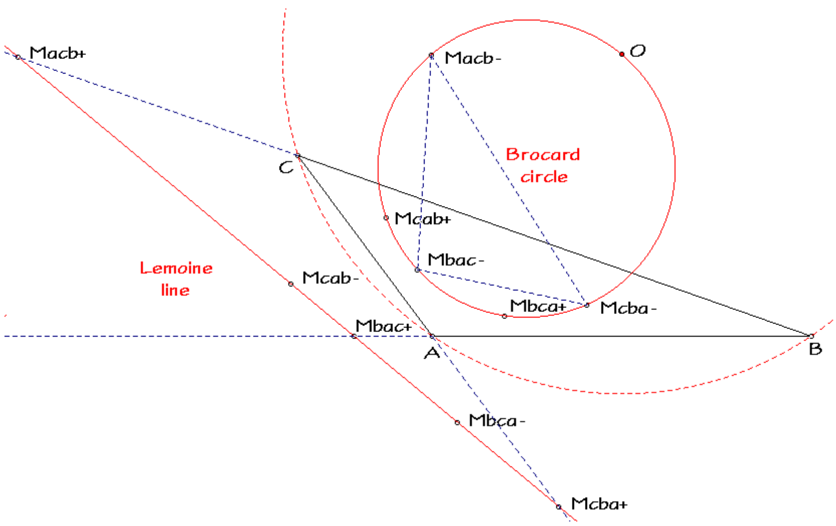


Fig. 6. Miquel points of *ABC*-similar inscribed triangles

Proof: (See also [1], S.1188) The barycentric coordinates of the Miquel points can be calculated with the conditions of their pedal triangles (see §2.2, §3):

$$M^+_{bca}(a^2b^2 : b^2c^2 : c^2a^2) \quad \text{and} \quad M^+_{cab}(a^2c^2 : b^2a^2 : c^2b^2)$$

are the Brocard points. The circumcenter $O$ and the Brocard points define the Brocard circle (Fig. 6) with the equation

$$a^4yz + b^4zx + c^4xy - a^2b^2z^2 - b^2c^2x^2 - c^2a^2y^2 = 0.$$

The Miquel points

$$M^-_{acb}(2S_A : b^2 : c^2),\ M^-_{cba}(a^2 : 2S_B : c^2),\ M^-_{bac}(a^2 : b^2 : 2S_C)$$

are points of the Brocard circle, too. More precisely, $M^-_{acb}, M^-_{cba}, M^-_{bac}$ are the vertices of the second Brocard triangle: The symmedian point

$$L(a^2 : b^2 : c^2)$$

lies diametrical to the circumcenter on the Brocard circle. The second points of intersection of its cevians and the Brocard circle define the second Brocard triangle. The Miquel points $M^-_{acb}, M^-_{cba}, M^-_{bac}$ on the Brocard circle are points of the cevians *LA, LB, LC* and therefore vertices of the second Brocard triangle.

The Miquel points

$$M^-_{bca}(a^2(a^2 - c^2) : b^2(b^2 - a^2) : c^2(c^2 - b^2)),$$

$$M^-_{cab}(a^2(a^2 - b^2) : b^2(b^2 - c^2) : c^2(c^2 - a^2))$$

are the inverses of the Brocard points in the circumcircle and

$$M^+_{acb}(0 : b^2 : -c^2),\ M^+_{cba}(a^2 : 0 : -c^2),\ M^+_{bac}(a^2 : -b^2 : 0)$$

are the midpoints of the Apollonius circles, all on the Lemoine line with the equation

$$a^2b^2z + b^2c^2x + c^2a^2y = 0.$$

The Lemoine line is the inverse of the Brocard circle with respect to the circumcircle. □

## 6. Equilateral inscribed triangles

For an equilateral pedal triangle $A''B''C''$ the introducing synthetic remark 2.2 gives the proportion

$$\text{for example} \quad \frac{AP}{BP} = \frac{b}{a}.$$

That means: $P$ is a point of the Apollonius circle for the sideline $AB$ of the given triangle (Fig. 7).
The three Apollonius circles have two common points, the isodynamic points *X(15)* and *X(16)* [5], which are the isogonal conjugates of the isogonic centers *X(13)* and *X(14)* and inverses with respect to the circumcircle.

**Theorem 4. The Miquel points of equilateral inscribed triangles are the isodynamic points. If the reference triangle isn´t equilateral, the midpoints lie on the tripolars of the isogonic centers.**

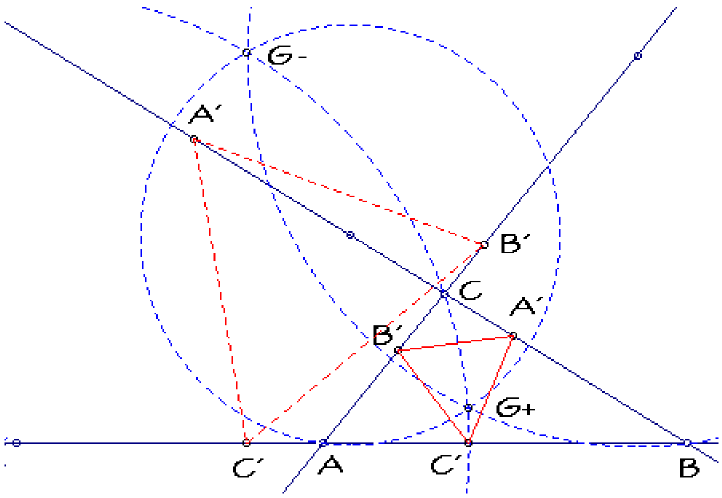

Fig. 7. Equilateral inscribed triangles

Proof: The pedal triangles of the isodynamic points

$$G^{\pm}(a^2(\pm S + \sqrt{3}S_A) : b^2(\pm S + \sqrt{3}S_B) : c^2(\pm S + \sqrt{3}S_C))$$

are equilateral but with different orientation ([1], S.1229). The sidelength is

$$\frac{S}{\sqrt{S_\omega \pm \sqrt{3}S}}.$$

Therefore $G^+$ and $G^-$ are the Miquel points of equilateral inscribed triangles with respect to the orientation. If the given triangle has an angle of 60° (120°), only one isodynamic point exists and the equilateral inscribed triangles have the same (a different) orientation as the given triangle.
For any real parameter κ we have two equilateral inscribed triangles $A'B'C'$:

$$A'_{\pm}(0 : b^2 + S_C \pm \sqrt{3}S - (\sqrt{3}S_A \pm S)\kappa : c^2 + S_B \pm \sqrt{3}S + (\sqrt{3}S_A \pm S)\kappa),$$
$$B'_{\pm}(a^2 + S_C \pm \sqrt{3}S + (\sqrt{3}S_B \pm S)\kappa : 0 : c^2 + S_A \pm \sqrt{3}S - (\sqrt{3}S_B \pm S)\kappa),$$
$$C'_{\pm}(a^2 + S_B \pm \sqrt{3}S - (\sqrt{3}S_C \pm S)\kappa : b^2 + S_A \pm \sqrt{3}S + (\sqrt{3}S_C \pm S)\kappa : 0)$$

with the sidelength $S\sqrt{\dfrac{1+\kappa^2}{S_w \pm \sqrt{3}S}}$.

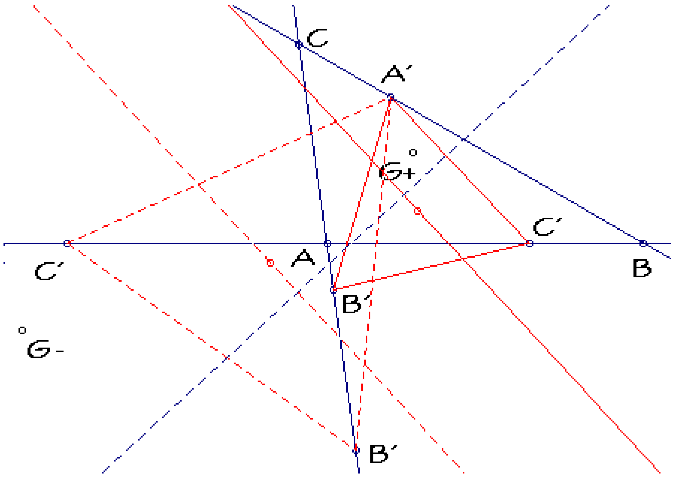


Fig. 8. Midpoints of equilateral inscribed triangles

The midpoints (see §3 for *α, β, γ*)

$$M((1+\alpha)(1+2\beta+\beta\gamma)$$
$$:(1+\beta)(1+2\gamma+\gamma\alpha)$$
$$:(1+\gamma)(1+2\alpha+\alpha\beta))$$

of the equilateral inscribed triangles lie on two parallel lines orthogonal to the Euler line of the reference triangle (Fig. 8). Their equations are

$$(\pm S+\sqrt{3}S_A)x+(\pm S+\sqrt{3}S_B)y+(\pm S+\sqrt{3}S_C)z=0.$$

Obviously these lines are the tripolars of the isogonic centers

$$F_{\pm}((\pm S+\sqrt{3}S_B)(\pm S+\sqrt{3}S_C)$$
$$:(\pm S+\sqrt{3}S_A)(\pm S+\sqrt{3}S_C)$$
$$:(\pm S+\sqrt{3}S_A)(\pm S+\sqrt{3}S_B))$$

and symmetrical with respect to the tripolar of the orthocenter.

□

## 7. Isosceles inscribed right triangles

Other special inscribed triangles are half-squares, that means isosceles right triangles. For each sideline which contains the vertex of the right angle there exists a Miquel point. With respect to the problem of inscribed squares for a quadrilateral, the position of the fourth square point of an inscribed half-square for a triangle is of interest.

If we remember the synthetic remark 2.1, the Miquel points must be the isogonal conjugates of points *K*, whose angles of sight for the sides are 90°, 135°, 135° (for positive orientation). Therefore we consider squares outside (+) and inside (}) of the given triangle with the midpoints $Q_a^{\pm}, Q_b^{\pm}, Q_c^{\pm}$ (Fig. 9):

e. g. $Q_a^{\pm}(-a^2 : S_C \pm S : S_B \pm S)$, $Q_c^{\pm}(S_B \pm S : S_A \pm S : -c^2)$.

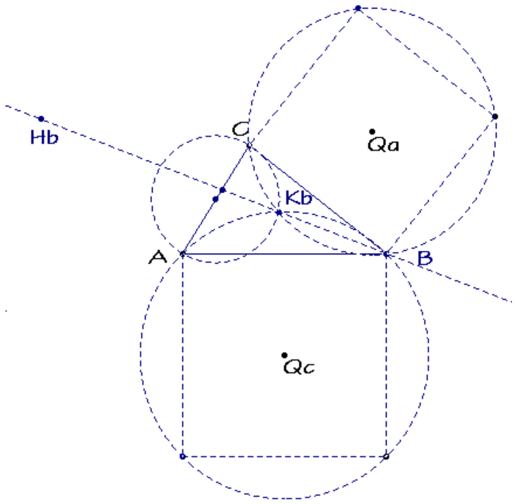

Fig. 9

The circumcircles of these squares have the second points of intersection $K_a^{\pm}, K_b^{\pm}, K_c^{\pm}$:

e. g. $K_b^{\pm}(S_B(\pm S + S_C) : (\pm S + S_A)(\pm S + S_C) : S_B(\pm S + S_A))$,

which are also points of the Thales circle over $CA$.
With these ***K*-points** we obtain:

**Theorem 5. For isosceles inscribed right triangles the Miquel points are the isogonal conjugates of the K-points.**
**The fourth square point of an inscribed half-square lies on the tripolar of the fourth harmonic point of the corresponding *K*-point on its cevian.**

Proof: The vertex of the right angle may be a point $B'$ on the sideline $CA$. Then the conditions

$$a'':b'':c'' \quad = \quad 1:\sqrt{2}:1$$

enable to calculate the Miquel points

$$M_b^{\pm}(a^2(\pm S + S_A) : b^2 S_B : c^2(\pm S + S_C)),$$

which are points of the Apollonius circle over $AC$ and symmetrical with respect to the circumcircle.

The coordinates show that the Miquel points are the isogonal conjugates of the $K$-points. The circumcircles of the side squares pass through two vertices of $ABC$, therefore their isogonal conjugates are circles too and we find the Miquel points in the second points of intersection of these circles (Fig. 10).
For every real parameter κ we obtain two isosceles inscribed right triangles (remember: the vertex of the right angle is a point of $AC$):

$$A'_{\pm}(0 : S_C \pm S - (S_A \pm S)\kappa : c^2 + S_B \pm S + (S_A \pm S)\kappa),$$

$$B'_{\pm}(a^2 \pm S + S_B\kappa : 0 : c^2 \pm S - S_B\kappa),$$

$$C'_{\pm}(a^2 + S_B \pm S - (S_C \pm S)\kappa : S_A \pm S + (S_C \pm S)\kappa : 0).$$

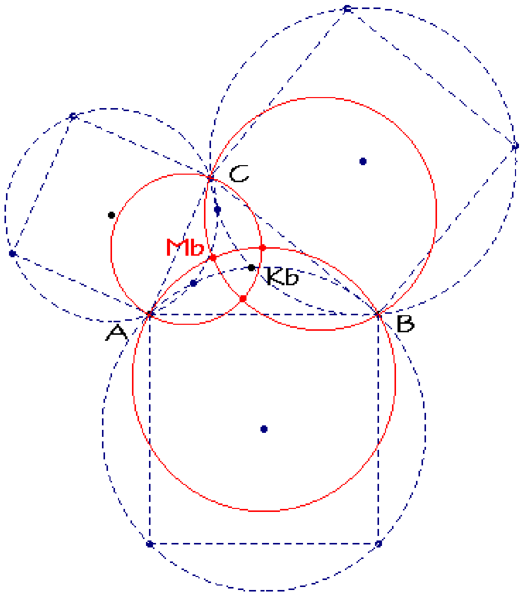


Fig. 10. Miquel point $M_b^+$ of isosceles inscribed right triangles

The length of the legs is

$$S\sqrt{\frac{1+\kappa^2}{S_B+S_w\pm 2S}}$$

and the fourth square point is

$$D'_{\pm}(S_B-\kappa(\pm S+a^2):\pm 2S+b^2-\kappa(S_A-S_C):S_B+\kappa(\pm S+c^2)).$$

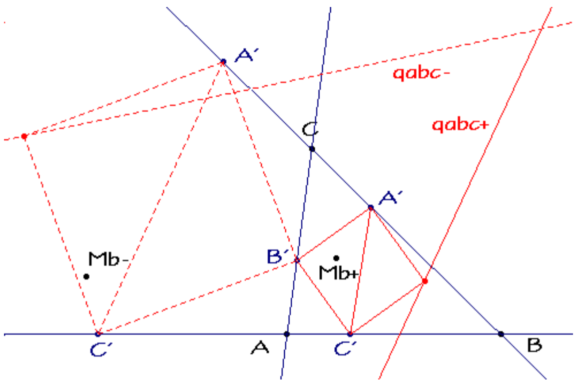


Fig. 11. Square-lines for the fourth square point

Elimination of $\kappa$ in the coordinates of the fourth square point gives the equations of two lines

$$q_{abc}^{\pm} \quad : \quad (\pm S+S_A)x-S_By+(\pm S+S_C)z=0.$$

These lines $q_{abc}^{\pm}$ for the fourth square point of inscribed half-squares may be called square-lines of *a, b, c* (Fig. 11). In Lamoen´s work ([6], 210) they are also to be found.

Considering the fourth harmonic points of $K_b^{\pm}$ on its cevian

$$H_b^{\pm}(S_B(\pm S+S_C):-(\pm S+S_A)(\pm S+S_C):S_B(\pm S+S_A)),$$

the tripolars obviously are the square-lines $q_{abc}^{\pm}$. □

Remark: Lamoen has treated inscribed squares for a triangle ([6], 208). Our results also enable a calculation. If the fourth square point D′ is for example a point of *AB*,

$$\kappa = -\frac{S_B}{c^2 + S}$$

gives an inscribed square with positive orientation:

$$A'(0 : S : c^2),\ B'(S : 0 : c^2),\ C'(S_B + S : S_A : 0),\ D'(S_B : S_A + S : 0)$$

and the sidelength $\dfrac{cS}{c^2 + S}$.

It is interesting to study the geometry of these Miquel points and square-lines more in detail. We only take into account the normal orientation (+).

The triangle of the Miquel points $M_a^+M_b^+M_c^+$ and the reference triangle are perspective, the perspector is the Kenmotu point *K=X(371):*

$$K(a^2(S + S_A) : b^2(S + S_B) : c^2(S + S_C)).$$

The Kenmotu point is the common vertex of the right angles of three congruent half-squares, whose other points lie on different sidelines of the reference triangle (Fig. 12).

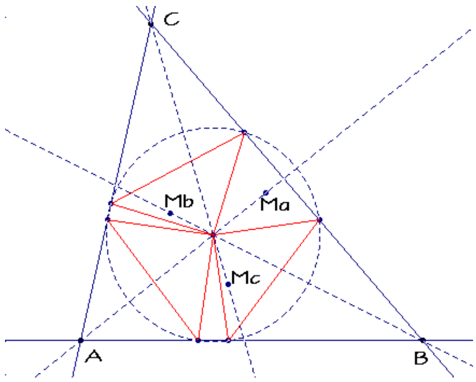


Fig. 12. The Kenmotu point

The Kenmotu point is the isogonal conjugate of the Vecten point *X(485)*

$$V((S + S_B)(S + S_C) : (S + S_A)(S + S_C) : (S + S_A)(S + S_B)),$$

which is the perspector of the reference triangle *ABC* and the triangle $Q_a^+Q_b^+Q_c^+$ of the midpoints of the squares outside *ABC*.

The triangle of the square-lines is also perspective with the reference triangle. The axis of perspective is the tripolar of the Vecten point.

A square-line can be constructed by special squares inscribed the triangle. This shall be shown in a nice geometric way (Fig. 13). Lamoen ([6], 207) gives the following construction of an inscribed square for a triangle: Construct an outside square for *AB* and connect the third and fourth square point with *C*. The points $C_a$ and $C_b$ of intersection with the sideline *AB* give a sideline of an inscribed square. $C_a$ is a point of $q_{cab}$, $C_b$ is a point of $q_{abc}$. The corresponding constructions for the other sides of the triangle give the square-lines $q_{cab}=B_aC_a$, $q_{abc}=A_bC_b$, $q_{bca}=A_cB_c$.

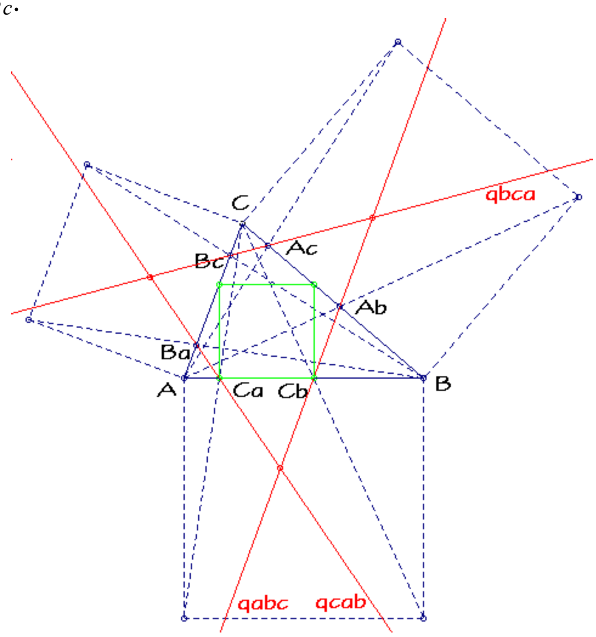


Fig. 13. A simple construction of the square-lines

The square-lines can be used to construct an inscribed square $A'B'C'D'$ for a quadrilateral with the sidelines $a, b, c, d$ (see also [6], S.212): $A'$ is the point of intersection of $a$ and the square-line $q_{bcd}$, $B'$ is the point of intersection of $b$ and the square-line $q_{cda}$, and further on (Fig. 14). If a square-line is parallel to the corresponding sideline, no inscribed square exists.

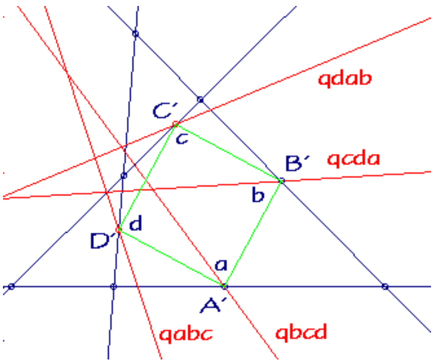


Fig. 14. Inscribed square of a quadrilateral

For an anti-clockwise orientation the side squares must be inside the given triangle. In analytical calculations the sign of $S$ has to be changed. The reflection in the circumcircle swaps the two triangles of Miquel points $M_a^+ M_b^+ M_c^+$ and $M_a^- M_b^- M_c$.

## 8. The general case

Till now we have looked for special inscribed triangles. Here we consider the pedal triangle of a point $P$ and all similar inscribed triangles with twelve Miquel points

$$P = M^+_{abc}, M^-_{abc}, M^\pm_{bca}, M^\pm_{cab}, M^\pm_{acb}, M^\pm_{cba}, M^\pm_{bac}$$

in the notation of §5.

**Theorem 6.** **The twelve Miquel points of inscribed triangles similar to the pedal triangle of a point *P* lie six by six on two circles invers with respect to the circumcircle and orthogonal to the Apollonius circles.**

$$P = M^+_{abc},\ M^+_{bca}, M^+_{cab}, M^-_{acb}, M^-_{cba}, M^-_{bac}$$

**are the reflections in the circumcircle of**

$$M^-_{abc},\ M^-_{bca}, M^-_{cab}, M^+_{acb}, M^+_{cba}, M^+_{bac}.$$

**The reflections in the Apollonius circles map Miquel points to Miquel points, so every circle through two vertices and a Miquel point passes another Miquel point:**

| | $M^{\pm}_{abc}$ | $M^{\pm}_{bca}$ | $M^{\pm}_{cab}$ |
|---|---|---|---|
| $M^{\mp}_{acb}$ | ***B, C*** | ***A, B*** | ***C, A*** |
| $M^{\mp}_{cba}$ | ***C, A*** | ***B, C*** | ***A, B*** |
| $M^{\mp}_{bac}$ | ***A, B*** | ***C, A*** | ***B, C*** |

For example: Reflecting the Miquel point $P = M^+_{abc}$ in the Apollonius circle over *BC,* we have the Miquel point $M^-_{acb}$ and the four points *B, C,* $M^+_{abc}$, $M^-_{acb}$ are concyclic.

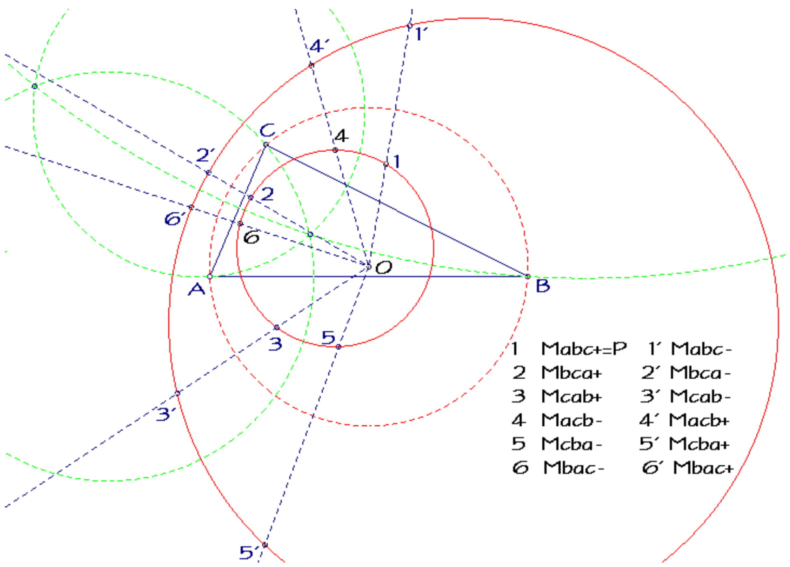


Fig. 15. The twelve Miquel points of similar inscribed triangles

Proof: The second statement is proved in theorem 1. For the first statement and the history of these so called Schoute circles see [1], 1230. In barycentric coordinates the Miquel points can be calculated from the modified conditions of the pedal triangle of *P* (see §3).

If we use normalized coordinates for *P* $(u+v+w=1)$ and denote the power of *P* with respect to the circumcircle with *p* (see below), we obtain the Miquel points with the following coordinates:

$$M_{abc}^{+}(u:v:w)=P,$$
$$M_{bca}^{+}(a^2[a^2b^2w-(c^2-a^2)p]$$
$$:b^2[b^2c^2u-(a^2-b^2)p]:c^2[a^2c^2v-(b^2-c^2)p]),$$
$$M_{cab}^{+}(a^2[a^2c^2v-(b^2-a^2)p]$$
$$:b^2[a^2b^2w-(c^2-b^2)p]:c^2[b^2c^2u-(a^2-c^2)p]),$$
$$M_{acb}^{-}(a^2b^2c^2u:b^2[a^2b^2w-(c^2-b^2)p]:c^2[a^2c^2v-(b^2-c^2)p]),$$
$$M_{cba}^{-}(a^2[a^2b^2w-(c^2-a^2)p]:a^2b^2c^2v:c^2[b^2c^2u-(a^2-c^2)p]),$$
$$M_{bac}^{-}(a^2[a^2c^2v-(b^2-a^2)p]:b^2[b^2c^2u-(a^2-b^2)p]:a^2b^2c^2w),$$

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$$M_{abc}^{-}(a^2[b^2c^2u+2S_Ap]:b^2[a^2c^2v+2S_Bp]:c^2[a^2b^2w+2S_Cp]),$$
$$M_{bca}^{-}(a^2[a^2b^2w+b^2p]:b^2[b^2c^2u+c^2p]:c^2[a^2c^2v+a^2p]),$$
$$M_{cab}^{-}(a^2[a^2c^2v+c^2p]:b^2[a^2b^2w+a^2p]:c^2[b^2c^2u+b^2p]),$$
$$M_{acb}^{+}(a^2[b^2c^2u+2S_Ap]:b^2[a^2b^2w+a^2p]:c^2[a^2c^2v+a^2p]),$$
$$M_{cba}^{+}(a^2[a^2b^2w+b^2p]:b^2[a^2c^2v+2S_Bp]:c^2[b^2c^2u+b^2p]),$$
$$M_{bac}^{+}(a^2[a^2c^2v+c^2p]:b^2[b^2c^2u+c^2p]:c^2[a^2b^2w+2S_Cp]).$$
$$M_{cba}^{+}(a^2[a^2b^2w-b^2p]:b^2[a^2c^2v-2S_Bp]:c^2[b^2c^2u-b^2p]),$$

The first six Miquel points lie on a Schoute circle, which can be described in the following way. Considering the ratio of the power of $X$ with respect to the circumcircle

$$pow(X)=-a^2yz-b^2zx-c^2xy \qquad (x+y+z=1)$$

and the distance of $X$ from the Lemoine line

$$lem(X)=\frac{S}{abc\sqrt{S_w^2-3S^2}}(a^2b^2z+b^2c^2x+c^2a^2y) \quad (x+y+z=1)$$

the circle contains all points $X$, for which this ratio is the same as for the point $P$:

$$\frac{pow(X)}{lem(X)}=\frac{pow(P)}{lem(P)}.$$

The equation of the first Schoute circle is

$$a^2yz+b^2zx+c^2xy=\mu(a^2b^2z+b^2c^2x+c^2a^2y)$$

$$\text{with} \quad \mu=\frac{a^2vw+b^2wu+c^2uv}{a^2b^2w+b^2c^2u+c^2a^2v},$$

if we use normalized coordinates for $X$ and $P$.

For the second Schoute circle, reflecting the first in the circumcircle, we have to replace $P=M_{abc}^{+}$ by $M_{abc}^{-}$.

The reflection in the Apollonius circle for example over $AB$ (midpoint $(a^2:-b^2:0)$ and radius $\frac{abc}{|a^2-b^2|}$) maps $M_{abc}^{\pm}$ to $M_{bac}^{\mp}$, $M_{bca}^{\pm}$ to $M_{acb}^{\mp}$, $M_{cab}^{\pm}$ to $M_{cba}^{\mp}$ . Therefore the Schoute circles are invariant under the reflection in the Apollonius circles and pass them orthogonal. □

Remarks:

(a) If $P$ is the circumcenter $O$:

$$pow(O) = \frac{a^2b^2c^2}{4S^2}, \quad lem(O) = \frac{abcS_w}{2S\sqrt{S_w^2 - 3S^2}}, \quad \mu = \frac{1}{2S_w},$$

we obtain the Miquel points of $ABC$-similar inscribed triangles (see §5). For other points of the Brocard circle the first Schoute circle is the Brocard circle, too.

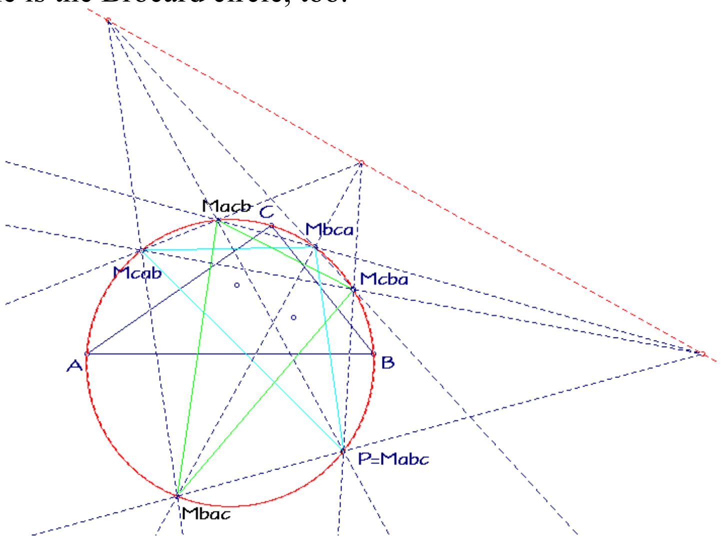


Fig. 16. Miquel points of similar Simson triangles

(b) If $P$ is a point of the circumcircle ($pow(P) = 0$) the pedal triangle degenerates on the Simson line, but we get six Miquel points on the circumcircle, which is the only Schoute circle. Considering the triangles $M_{abc}M_{bca}M_{cab}$ and $M_{acb}M_{cba}M_{bac}$ they are perspective with respect to the midpoints of the Apollonius circles and have the same Brocard points as the reference triangle (Fig. 16).

## 9. Similar circumscribed triangles

For circumscribed triangles $A'B'C'$ the reference triangle $ABC$ is an inscribed triangle with $A$ on $B'C'$, $B$ on $C'A'$, $C$ on $A'B'$. We look for the Miquel points of the reference triangle with respect to similar circumscribed triangles.

**Theorem 7. If $P$ is the Miquel point of similar inscribed triangles, the isogonal conjugate $P^*$ is the Miquel point of the reference triangle with respect to all similar circumscribed triangles.**

Proof: The pedal triangle $A''B''C''$ of the Miquel point $P$ is orthologic to the reference triangle (see §2.1). If we draw parallels to the sidelines of $A''B''C''$ through $A$, $B$, $C$, we get a circumscribed triangle $A'B'C'$ similar to $A''B''C''$. The Miquel point of the reference triangle with respect to $A'B'C'$ and all similar circumscribed triangles obviously is $P^*$ (Fig. 17). □

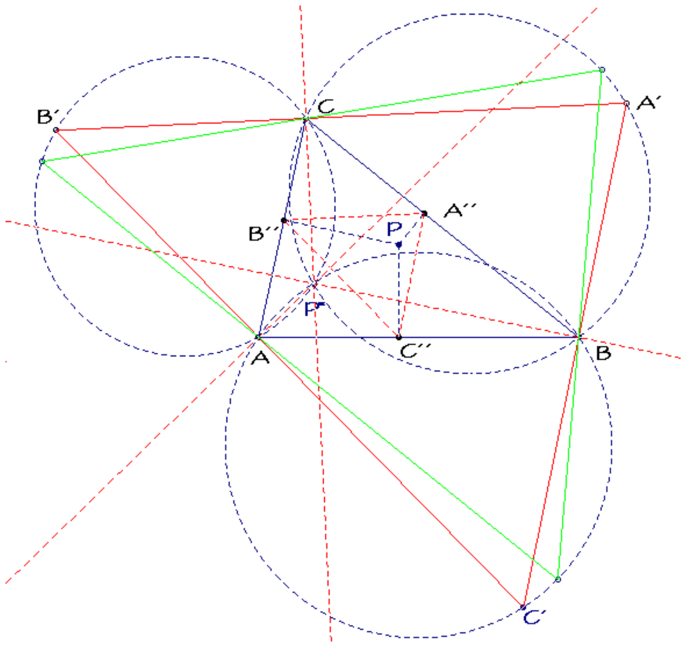


Fig. 17. Similar circumscribed triangles

We get the Miquel points $N_{ijk}^{\pm}$ of the reference triangle with respect to similar circumscribed triangles as the isogonal conjugates of the Miquel points $M_{ijk}^{\pm}$ of corresponding similar inscribed triangles. The reflection in the circumcircle (°) maps $M_{ijk}^{+}$ to $M_{ijk}^{-}$, the operation $X \to X^{*o*}$ maps $N_{ijk}^{+}$ to $N_{ijk}^{-}$. The Miquel points $M_{ijk}^{\pm}$ lie on two circles, that doesn´t hold for the Miquel points $N_{ijk}^{\pm}$, but a circle through two vertices of the reference triangle and a Miquel point $N_{ijk}^{\pm}$ passes another one (see theorem 6).

For equilateral circumscribed triangles the Miquel points of the reference triangle are the isogonic centers (see §2.1 and §6), for isosceles circumscribed right triangles we have the K-points (see §7). More interesting are circumscribed *ABC*-similar triangles (Fig. 18):

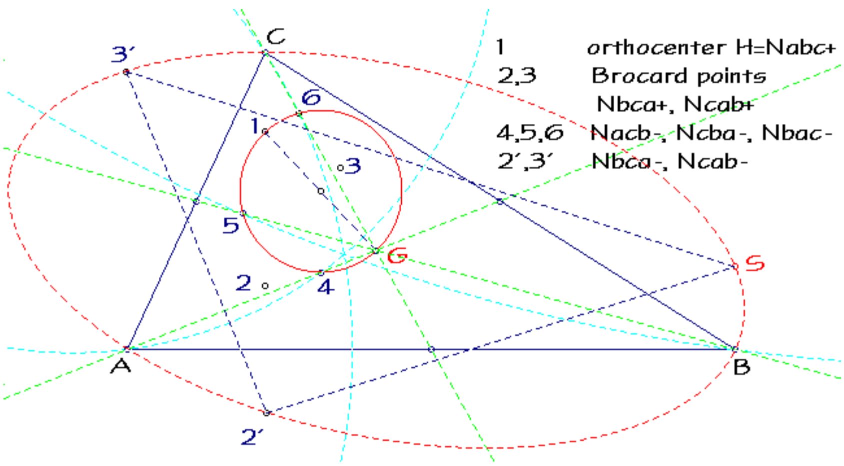


Fig. 18. Miquel points of circumscribed *ABC*-similar triangles

$N^+_{abc}$ is the orthocenter $H$.

$N^+_{bca}$, $N^+_{cab}$ are the Brocard points again.

$N^-_{acb}(a^2 : 2S_A : 2S_A)$, $N^-_{cba}(2S_B : b^2 : 2S_B)$, $N^-_{bac}(2S_C : 2S_C : c^2)$

are the points of intersection of the medians and the Apollonius circles. They are concyclic with the orthocenter $H$ and the centroid $G$ and $HG$ is a diameter of this circle.

$N^-_{abc}$ doesn´t exist.

$N^-_{bca}((a^2-b^2)(b^2-c^2) : (b^2-c^2)(c^2-a^2) : (c^2-a^2)(a^2-b^2))$,

$N^-_{cab}((a^2-c^2)(c^2-b^2) : (b^2-a^2)(a^2-c^2) : (c^2-b^2)(b^2-a^2))$

are points of the Steiner ellipse, which is the isogonal conjugate of the Lemoine line. A third point on the Steiner ellipse is the Steiner point

$$S((a^2-b^2)(a^2-c^2) : (b^2-c^2)(b^2-a^2) : (c^2-a^2)(c^2-b^2)).$$

The triangle $N^-_{bca}N^-_{cab}S$ has the same centroid and the same area as the reference triangle. Further on there is a relation to the first Brocard triangle $A°B°C°$, whose vertices are the second points of intersection of the Brocard circle and the perpendicular bisectors of the sides. Parallels to the sidelines of the first Brocard triangle through the vertices of the reference triangle give the points $N^-_{bca}$, $N^-_{cab}$, $S$:

| parallel | $A$ | $B$ | $C$ |
|---|---|---|---|
| $A°B°$ | $N^-_{bca}$ | $N^-_{cab}$ | $S$ |
| $B°C°$ | $S$ | $N^-_{bca}$ | $N^-_{cab}$ |
| $C°A°$ | $N^-_{cab}$ | $S$ | $N^-_{bca}$ |

$N^+_{acb}$, $N^+_{cba}$, $N^+_{bac}$ don´t exist.

## 10. Cevian triangles and their Miquel points

Last but not least let us have another view of inscribed triangles and their Miquel points. Let $M_P$ be the Miquel point of the cevian triangle of a point $P(u:v:w)$. The vertices of the cevian triangle divide the sides in the ratios

$$\alpha = \frac{w}{v}, \ \beta = \frac{u}{w}, \ \gamma = \frac{v}{u}.$$

Hence the Miquel point is (see §3)

$$M_P\Big(\frac{a^2}{v+w}\Big(\frac{vwa^2}{v+w} - \frac{wub^2}{w+u} - \frac{uvc^2}{u+v}\Big)$$

$$: \frac{b^2}{w+u}\Big(-\frac{vwa^2}{v+w} + \frac{wub^2}{w+u} - \frac{uvc^2}{u+v}\Big) : \frac{c^2}{u+v}\Big(-\frac{vwa^2}{v+w} - \frac{wub^2}{w+u} + \frac{uvc^2}{u+v}\Big)\Big)$$

In this way we get a transformation, which maps a point $P$ in the Miquel point $M_P$ of its cevian triangle.
For example: The orthocenter is a fix point, the centroid becomes the circumcenter and the Gergonne point becomes the incenter (see also [7], 211).

| $P$ | X(2) | X(4) | X(7) | X(8) | X(69) | X(189) | X(253) |
|---|---|---|---|---|---|---|---|
| $M_P$ | X(3) | X(4) | X(1) | X(40) | X(20) | X(84) | X(64) |

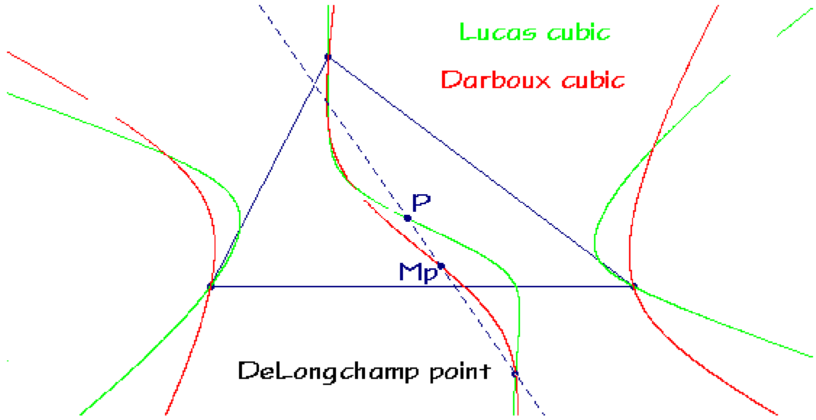

Fig. 19. Lucas and Darboux cubic

All points whose cevian triangle are pedal triangles lie on the Lucas cubic (Fig. 19) with the equation [3]

$$\sum_{cyclic} S_A x(y^2 - z^2) = 0,$$

which is isotomic invariant with Pivot *X(69)* (isotomic conjugate of the orthocenter). The corresponding Miquel points are on the Darboux cubic (Fig. 19) with the equation [3]

$$\sum_{cyclic} (S^2 - 2S_B S_C) x(c^2 y^2 - b^2 z^2) = 0,$$

which is isogonal invariant with Pivot X(20) (DeLongchamp point).
The connecting lines $PM_P$ for points $P$ on the Lucas cubic and $M_P$ on the Darboux cubic all pass through the DeLongchamp point. For isotomic conjugates on the Lucas cubic the corresponding Miquel points are symmetric with respect to the circumcenter on the Darboux cubic (see also [4], 142).

A last remark for anticevian triangles: Let $N_P$ be the Miquel point of the reference triangle with respect to the anticevian triangle of a point $P(u:v:w)$:

$$N_P\left(\frac{u(-u+v+w)}{-a^2vw+b^2wu+c^2uv} : \frac{v(u-v+w)}{a^2vw-b^2wu+c^2uv} : \frac{w(u+v-w)}{a^2vw+b^2wu-c^2uv}\right)$$

For example: The anticevian triangle of the Lemoine point *L=X(6)* is the tangential triangle of *ABC*. Hence the circumcenter O=*X(3)* is the Miquel point of *ABC* with respect to its tangential triangle (Fig. 20).

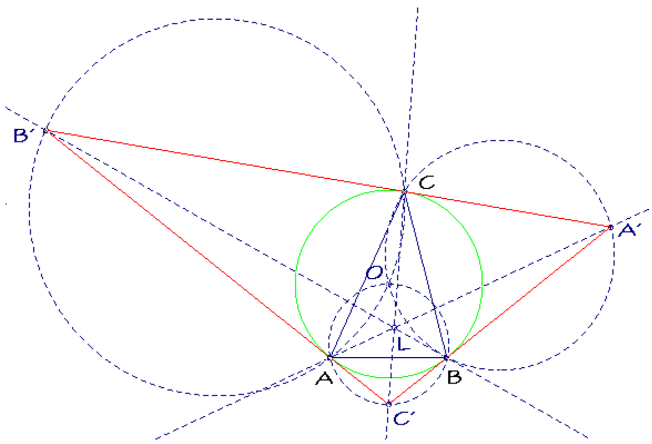


Fig. 20. The tangential triangle

| $P$ | X(1) | X(2) | X(3) | X(4) | X(6) | X(9) | X(57) |
|---|---|---|---|---|---|---|---|
| $N_P$ | X(1) | X(4) | X(64) | X(20) | X(3) | X(84) | X(40) |

All points of the first row lie on the Thomson cubic with the equation [3]:

$$\sum_{cyclic} S_A x(y^2 - z^2) = 0,$$

which is isogonal invariant with Pivot $G$ (centroid). The Thomson cubic is the locus of points $P$ whose anticevian triangle is orthologic to $ABC$. The Miquel points $N_P$ are again on the Darboux cubic [3].

The connecting lines $PN_P$ for points $P$ on the Thomson cubic and $N_P$ on the Darboux cubic all pass through the circumcenter. For isogonal conjugates $P$ and $P^*$ on the Thomson cubic the corresponding Miquel points on the Darboux cubic are isogonal conjugates, too.

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